OPTIMAL HARVESTING OF A RENEWABLE RESOURCE: A MATHEMATICAL MODEL

Sudipto Sarkar
DeGroote School of Business, McMaster University, Hamilton, ON L8S 4M4, Canada.
Corresponding author: sarkars@mcmaster.ca

Available at http://www.ssrn.com/link/OIDA-Intl-Journal-Sustainable-Dev.html

Abstract: This paper derives the optimal harvesting policy for a renewable natural resource using a real-option model. The harvesting policy is illustrated using the well-known "tree-cutting" or "optimal rotation" problem from Forest Economics. We show how to identify the optimal tree size at which to harvest (or the optimal harvesting trigger), and also the resulting expected rotation period. The optimal trigger depends on (i) the growth dynamics of the tree, (ii) the interest rate, (iii) the harvesting cost, and (iv) size of the replacement seedling. With reasonable input parameters, it is shown that tree size must increase to 4½ times the original size before it is optimal to harvest it, which gives an expected rotation period of 51 years. The model is flexible enough to accommodate various adjustments if necessary.

Keywords: Renewable resource; Harvesting; Optimal policy; Mathematical model.

Introduction

Many renewable natural resources are under serious threat. For instance, a number of fishing areas have become significantly depleted, threatening future stocks and increasing the uncertainty regarding fishing productivity (Murillas and Chamorro, 2006, Clark, 1990, Conrad, 2008, etc). However, if properly managed, renewable resources such as fish and forest stocks should not really be at risk of extinction (Sarkar, 2009). This is because these resources grow or multiply over time. Thus, these shortages can be avoided with proper management. But "proper management" means different things to different people. For instance, biologists or conservationists might prefer continuous harvesting so as to ensure the maximum sustainable yield, while economists might prefer discrete or "pulse" harvesting because of fixed costs of harvesting.

While the conservationists' approach might be appropriate in an ideal world, in the real world one must take into account such important economic factors as markets, interest rates, financing costs, harvesting costs, future growth rate, etc. For instance, an economic reason for higher harvesting rates (leading to a reduction in resource level below the maximum sustainable yield) is to encourage faster growth, which is a feature of mean-reverting growth models that approximate some renewable resource growth patterns.

This paper proposes a general economic model of renewable resource harvesting that takes into account the salient features mentioned above – the effect of markets, harvesting cost, time value of money (interest rate), etc. Also, very importantly, we build a perpetual model that is forward-looking; thus, it takes into account future growth, and thereby forces us to take a more long-term view. This encourages long-term thinking and makes short-term harvest maximization less attractive. Finally, the model incorporates biological risks that often have a significant impact on the size of renewable resources, e.g., risk of fire or disease.

We show how to optimally manage the renewable resource stock, so as to maximize the total well-being as well as bring down the risk of extinction to negligible levels. We use a real-option model, which has recently become popular in the Economics literature as a way to model investment decisions.

In traditional models of renewable resource harvesting, the steady-state rate of harvest equals the rate of growth so that the stock level is maintained at the optimal level (Clark, 1990). This might be good in an ecological sense (since it reduces the risk of extinction), but not necessarily in an economic sense for the following reasons: (i) in the presence of fixed costs, continuous harvesting can prove very costly; because of fixed costs, it makes economic sense to harvest large quantities (which will obviously lower the stock level) and then allow regeneration, which our model allows; and (ii) growth is faster when the stock level is lower, while harvesting is more efficient when the stock level is higher; our model takes advantage of both these factors by harvesting when stock level is high enough and reducing the level which will speed up growth.

Thus, our real-option model has three advantages over traditional steady-state models: (a) growth rate is higher because the stock level is reduced by harvesting a larger quantity (in spite of which the risk of extinction is generally negligible), (b) harvesting is done when the stock level is higher, thus making the harvest more efficient and less expensive, and (c) because of fixed costs of harvesting, larger discrete harvests are more economically efficient than continuous harvesting.

In our model, the optimal harvesting policy specifies the harvest trigger, and the resulting forest rotation (in expectation, because of uncertainty). The optimal policy maximizes the value of the resource stock. We also carry out some sensitivity analysis to see how the optimal harvesting policy and expected rotation change when the input parameters are varied. Comparative static results indicate that the optimal harvest trigger is an increasing function of the growth rate and volatility of tree size dynamics as well as the harvesting cost; and a decreasing function of interest rate and value of replacement seedling. The expected rotation is an increasing function of volatility and harvesting cost, and a decreasing function of growth rate, interest rate and value of replacement seedling. Thus, when there is greater uncertainty regarding the tree growth dynamics, harvesting must be delayed, and harvests should occur less frequently. That is, not surprisingly, a higher level of uncertainty causes the optimal policy to be more cautious. Also, faster-growing trees should be harvested more frequently, although they should be allowed to reach a larger size before being cut.

The rest of the paper is organized as follows. Section 2 describes the model and derives the optimal harvesting and the corresponding expected rotation period. Section 3 discusses and summarizes the results, both analytical and numerical. Section 4 concludes.

The Model

Specifications

We illustrate the model using the well-known “tree-cutting” problem, also known as the “optimal rotation problem” in Forest Economics. However, although the particular problem is set in the area of forest economics, the issues involved in the model are quite broad in scope, particularly in other renewable resource areas such as fisheries where uncertainty in an important factor. A major component of our model is the derivation of the optimal harvesting rule in an uncertain setting, which consists of an “optimal harvesting trigger.”

Let x_t be the size of a stand of trees (or a forest). This is the state variable in the model, and changes randomly reflecting the uncertainties inherent in natural resource values. We assume that x_t evolves according to the general stochastic process:

$$dx = \mu(x)dt + \sigma(x)dz \tag{1}$$

Equation (1) states that the instantaneous change in the value of the tree stand is given by the sum of two components. The first component captures the deterministic part of the growth; thus, the value is expected to increase at a rate of $\mu(x)$ over the next instant. The second component captures the uncertainties or the stochastic nature of natural resources. Thus, in addition to the expected increase in value, there is a random term $\sigma(x)dz$, where dz is the increment of a standard Brownian Motion Process (BMP), which is a commonly-used process in denoting randomness or uncertainty. Here, $\sigma(x)$ is a measure of the uncertainty or volatility of the value growth process. Thus, the terms $\mu(x)$ and $\sigma(x)$ represent the expected growth rate and the volatility, respectively, of the growth process for the value of the tree stand. At this point, both are general functions of the state variable x, but we will have to specify them more explicitly when we solve the model. Finally, the discount rate is r.

The harvesting policy is as follows: when x_t rises sufficiently, say to a level U, the tree stand is harvested and sold for an amount \$U. However, there is a harvesting cost \$c, hence the net proceeds from harvesting is \$(U – c). When harvested, the trees are replaced with seedlings, whose value is \$L. Thus, the (controlled) state variable x falls from the upper limit U to the renewal level L just after harvest.[1]

Our model differs somewhat from the usual approach in the real-option literature on resource harvesting (Chang, 2005, Shackleton and Sodal, 2010). This is because, unlike the existing papers, we use a differential equation to solve for the value of the renewable resource and identify the optimal policy from the appropriate boundary condition. The value of the resource (here forest or stand of trees) will obviously be a function of the state variable x. Since we (in common with all papers on the rotation problem) assume an infinite horizon, the value function will be independent of time. We therefore write the value of the resource as V(x).

Valuation

It can be shown that V(x) must satisfy the ordinary differential equation (ODE):

$$0.5[\sigma(x)]^2 V''(x) + \mu(x)V'(X) - rV(X) = 0 \qquad (2)$$

In order to explicitly solve this ODE, there needs to be more structure on the functions $\mu(x)$ and $\sigma(x)$. For the well-known lognormal process, the ODE (2) has an analytical solution. Therefore, to illustrate the solution of the model, let us assume that x follows a lognormal process, as in Chang (2005) and Shackleton and Sodal (2010). Then we have $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$. With this specification, the general solution to ODE (2) is:

$$V(x) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2} \qquad (3)$$

where A_1 and A_2 are constants to be determined by the boundary conditions, and γ_1 and γ_2 are the positive and negative solutions, respectively, to the quadratic equation

$$0.5\sigma^2\gamma(\gamma - 1) + \mu\gamma - r = 0$$

and are given by

$$\gamma_1 = 0.5 - \mu/\sigma^2 + \sqrt{2r/\sigma^2 + \left(0.5 - \mu/\sigma^2\right)^2} \quad > 1 \qquad (4)$$

$$\gamma_2 = 0.5 - \mu/\sigma^2 - \sqrt{2r/\sigma^2 + \left(0.5 - \mu/\sigma^2\right)^2} \quad < 0 \qquad (5)$$

When $x \rightarrow 0$ (i.e., falls to very low levels), the value function V(x) will also approach zero, since x = 0 is an absorbing boundary for a lognormal process. This implies $A_2 = 0$ in equation (3). Thus, the value function becomes:

$$V(x) = A_1 x^{\gamma_1} \qquad (6)$$

Boundary Conditions

When x rises to the level U, the stand is harvested and replaced with seedlings, generating a cash flow of (U – c). Thus, the state variable will go from U to L (as in the other papers in this literature, we assume the harvesting/replanting is done instantaneously). Just after replanting, then, the value of the stand will be V(L). Thus, the payoff at harvest is {U – c + V(L)}This gives the value-matching boundary condition:

$$V(U) = U - c + V(L) \qquad (7)$$

[1] The replanting cost is included in the harvesting cost c. Since replanting includes the acquisition of the seedlings, it must be the case that c > L.

The other boundary condition is the smooth-pasting condition, which requires that the derivative of the value at harvesting be equal to the derivative of the payoff. This is the optimality condition, which ensures that the harvest trigger U is optimal (Dixit and Pindyck, 1994). Thus, the second boundary condition is:

$$\left.\frac{dV(x)}{dx}\right|_{x=U} = \frac{\partial(\text{Payoff})}{\partial U} \tag{8}$$

The value-matching and smooth-pasting conditions simplify, respectively, to:

$$A_1 U^{\gamma_1} = U - c + A_1 L^{\gamma_1} \tag{9}$$

$$A_1 \gamma_1 U^{\gamma_1 - 1} = 1 \tag{10}$$

The smooth-pasting condition gives the constant A_1 in terms of the trigger U:

$$A_1 = \frac{U^{1-\gamma_1}}{\gamma_1} \tag{11}$$

Substituting into the value-matching condition, we get an equation that can be solved for the optimal harvest trigger U:

$$f(U) = L^{\gamma_1} U^{1-\gamma_1} + (\gamma_1 - 1)U - \gamma_1 c = 0 \tag{12}$$

Thus, although it is not possible to derive an analytical expression for the optimal harvest trigger, it can be very easily computed using numerical techniques. It is also easy to prove that a unique optimal trigger exists.

Finally, since a major point of interest in this literature is the "optimal rotation" or the period between harvests, we would like to know the expected optimal rotation for our model, i.e., the expected time between consecutive harvests, or the expected time for x to go from L to U.[2] For a lognormal process with the drift and diffusion parameters μ and σ respectively, this expected time is given by:[3]

$$E(T) = \frac{\ln(U/L)}{\left(\mu - 0.5\sigma^2\right)} \tag{13}$$

This gives us the optimal rotation (in expectation) for the model.

Results

Analytical Results

The analytical results are discussed in this section, followed by numerical results in Section 3.2. The first result deals with the existence and uniqueness of the optimal harvest trigger.

Proposition 1.

The optimal harvest trigger is the solution to equation (12). Although there is no analytical solution to this equation, the optimal harvest trigger does exist and is unique.

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[2] Our model has an expected optimal rotation rather than an optimal rotation because of the stochastic nature of our model, in contrast to traditional models such as Binkley (1987) or Conrad (2008).

[3] See Mauer and Ott (2000) for a derivation of the result.

Proof: Substituting into equation (12), we get $f(L) = \gamma_1(L - c)$, which is negative since $c > L$ and $\gamma_1 > 1$. Also, we get $\lim_{U \to \infty} f(U) = +\infty$. Therefore, for some U between L and ∞, f(U) must be zero; hence a solution to f(U) = 0 must exist. To show that it is a unique solution, we differentiate f(U) with respect to U to get:

$$\frac{df}{dU} = (\gamma_1 - 1)\left[1 - (L/U)^{\gamma_1}\right] \tag{14}$$

Since $\gamma_1 > 1$ and $U > L$, df/dU will always be positive. Since f(L) is negative and $\lim_{U \to \infty} f(U) = +\infty$ and df/dU is always positive, there can be only one solution to the equation f(U) = 0. Thus, the solution is unique.

Q.E.D.

Proposition 1 is important because it establishes the existence and uniqueness of the optimal harvesting trigger, even though the optimal trigger can be computed only numerically and not analytically. Next, we establish the effect of the replacement size L on the optimal harvest trigger, in Proposition 2 below.

Proposition 2.

The harvest trigger is a decreasing function of the replacement seedling size L.

Proof: Differentiating equation (12) with respect to L, we get after some simplification:

$$\frac{dU}{dL} = -\frac{\gamma_1 (L/U)^{\gamma_1 - 1}}{(\gamma_1 - 1)\left[1 - (L/U)^{\gamma_1}\right]} \tag{15}$$

Since $\gamma_1 > 1$ and $U > L$, dU/dL will always be negative.

Q.E.D.

Numerical Results

Base Case

Since equation (12) has to be solved numerically for the optimal harvest trigger U, we need to specify the input parameter values to illustrate the results. While we choose the "base-case" parameter values somewhat arbitrarily, the numerical solutions are repeated with a wide range of parameter values, and the qualitative results of the model are found to be valid irrespective of the parameter values.

For the base case, as in Shackleton and Sodal (2010), we set the replanting size at L = 1, which is basically a normalization. For the cost of harvesting and replanting, we assume that it is 50% higher than the seedling cost, hence c = 1.5. The discount rate is set at r = 9%. Finally, for the tree growth dynamics, we set the expected growth rate and volatility at μ = 6% and σ = 25% respectively.

With these input values, solving equation (12) gives U = 4.3692. Thus, it is optimal to wait till the tree has grown to almost four and a half times the initial size before harvesting. From equation (13), the expected time for x to go from L to U comes to 51.29 years; thus, the expected optimal rotation is 51.29 years. The actual rotation could, of course, be different, depending on the actual evolution of x over time.

Comparative Statics

Table 1 shows the results when the solution is repeated with a wide range of input parameter values. The comparative static results are summarized in:

Result 1.

(a) The harvest trigger U is an increasing function of the parameters μ, σ and c, and a decreasing function of r and L.
(b) The expected rotation is an increasing function of the parameters c and σ, and a decreasing function of μ, r and L.

The important comparative static results are discussed below. First, the harvest trigger U falls as L is increased; thus, if the replacement seedling size is increased, the resulting trees are harvested at a smaller size or more frequently.

Table 1. Optimal Harvest Policy and Optimal Rotation

Base case values: r = 9%, μ = 6%, σ = 25%, L = 1, and c = 1.5.

Optimal harvest trigger: U = 4.3692. Optimal expected rotation E(T) = 51.29 years.

Comparative statics:

L	U	E(T)
0.8	4.9736	63.56
0.9	4.6840	57.37
1.0	4.3692	51.29
1.1	4.0241	45.11
1.2	3.6392	38.59

c	U	E(T)
1.3	3.3129	41.66
1.4	3.8511	46.90
1.5	4.3692	51.29
1.6	4.8737	55.09
1.7	5.3684	58.45

r	U	E(T)
0.07	8.1049	72.78
0.08	5.3623	58.41
0.09	4.3692	51.29
0.10	3.8407	46.81
0.11	3.5069	43.64

μ	U	E(T)
0.04	3.2499	134.70
0.05	3.6829	69.53
0.06	4.3692	51.29
0.07	5.6577	44.72
0.08	9.2177	44.56

σ	U	E(T)
0.23	4.2398	43.06
0.24	4.3036	46.78
0.25	4.3692	51.29
0.26	4.4366	56.87
0.27	4.5056	63.92

This might seem a puzzling result, but it is driven by the assumption that the harvesting cost c is constant. Since c is unchanged when L is increased, this assumption implies that, for the same cost, we are getting seedlings worth more at each harvest. Then it makes sense to increase the frequency of harvesting, hence U is lower and expected rotation is shorter.

Next, when the discount rate r is lower, there is less incentive to harvest since the present value does not decline sharply; hence the harvest trigger U is higher. Conservationists might prefer this outcome, as it leaves the forest less vulnerable to unexpected events such as forest fires.

When the expected growth rate μ is higher, it becomes more worthwhile to wait since the trees are expected to grow faster, hence the trigger U is higher. However, this does not mean the harvesting is less frequent, because although the trigger is higher the higher growth rate actually reduces the expected time to reach the trigger, thereby shortening the expected rotation. This gives a negative relationship between growth rate and expected rotation length.

A higher volatility σ will increase the harvest trigger, which is a basic result from option theory, since the value of waiting increases. The other results are intuitively obvious.

Conclusion

This paper uses a real-option model to identify the optimal harvesting policy for a renewable natural resource such as a forest or a tree stand. The optimal policy is derived numerically, but we prove rigorously the existence and uniqueness of the optimal policy. Numerical solutions indicate that the model generates reasonable results, e.g., an expected rotation of around 50 years.

This version of our paper is work-in-progress. We would like to extend it in the following (and possibly other) ways. First, we have assumed (as in other papers in the literature, e.g., Chang, 2005, Shackleton and Sodal, 2010) that the harvesting cost c is constant, irrespective of the levels of L and U. In real life, this is unlikely to be correct, because the replacement cost component of total harvesting cost will increase directly with the size of seedlings used (L). Also, since it takes more effort to harvest a larger tree than a smaller one, harvesting cost should be an increasing function of the harvest trigger U. In addition, there should probably be a fixed cost component that is independent of L and U. We would like to extend the current model by using a harvesting cost function that has three components: (i) a fixed cost, (ii) increasing function of L, and (iii) increasing function of U. This could potentially make a significant difference to the results because the boundary conditions (particularly the smooth-pasting) would be different.

Second, we would like to also look at different growth processes for the tree size dynamics. While we have used the lognormal process for simplicity (following Chang, 2005, and Shackleton and Sodal, 2010), some researchers have suggested that a mean-reverting stochastic process could also be appropriate here.

Finally, our model assumes that the replanting level L is exogenously specified, and the owner of the forest has no say regarding its magnitude. However, if the owner/operator can choose L, what would be the optimal level of L? We plan to extend the current model by addressing this question.

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